14. Revisão e aprofundamento

Lembrete 14.0.1 Sejam $f : [0,L] \to \mathbb{R}$ uma função, e $f_p(x)$, $f_i(x)$ extensões par e ímpar de f.

(1) Série de senos de f é

$$S[f_i] = \sum_{n=1}^{\infty} b_n \operatorname{sen}\left(\frac{\pi n x}{L}\right),$$

onde

$$b_n = \frac{1}{L}\int_{-L}^{L} f_i(x) \operatorname{sen}\left(\frac{\pi n x}{L}\right) dx = \frac{2}{L}\int_{0}^{L} f(x) \cdot \operatorname{sen}\left(\frac{\pi n x}{L}\right) dx.$$

(2) Série de cossenos de f é

$$S[f_p] = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi n x}{L}\right),$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f_p(x) dx = \frac{2}{L}\int_{0}^{L} f(x) dx,$$
$$a_n = \frac{1}{L}\int_{-L}^{L} f_p(x) \cos\left(\frac{\pi n x}{L}\right) dx = \frac{2}{L}\int_{0}^{L} f(x) \cdot \cos\left(\frac{\pi n x}{L}\right) dx.$$

■ **Exemplo 14.1** Ache a soma $S(x)$ da série de senos de $f(x) = x^2$, $0 \le x \le \pi$ para todo x. Calcule $S(\frac{45\pi}{4})$, $S(37\pi)$ e $S(\frac{2014\pi}{3})$.

Solução. Usando Exemplo 13.5 temos que $S[f_i](x) = f_i(x)$ se $x \in (-\pi,\pi)$ e $S[f_i](\pm\pi) = 0$. Além disso $S[f_i]$ é 2π-periódica, portanto

$$S(x) = S[f_i](x) = \begin{cases} (x-2\pi n)^2, & 2\pi n \le x < \pi + 2\pi n, \\ -(x-2\pi n)^2, & -\pi + 2\pi n < x < 2\pi n, \\ 0, & x = \pi + 2\pi n,\ n \in \mathbb{Z}. \end{cases}$$

Logo

$$S(\frac{45\pi}{4}) = S(\frac{44\pi}{4} + \frac{\pi}{4}) = S(\pi + \frac{\pi}{4}) = S(\pi + \frac{\pi}{4} - 2\pi) = S(-\frac{3\pi}{4}) = \frac{-9\pi^2}{16}.$$

$$S(37\pi) = S(36\pi + \pi) = S(\pi) = 0.$$

$$S(\frac{2014\pi}{3}) = S(671\pi + \frac{\pi}{3}) = S(670\pi + \pi + \frac{\pi}{3}) = S(\pi + \frac{\pi}{3} - 2\pi) = S(-\frac{2\pi}{3}) = -\frac{4\pi^2}{9}.$$

;-)

■ **Exemplo 14.2** Ache a série de cossenos para $f(x) = x$, $x \in [0,1]$. Ache a soma da série de cossenos $S(x)$ para todo x e esboce o gráfico dela. Calcule $S(2015)$, $S(-\frac{2013}{2})$.

Solução.

$$f_p(x) = \begin{cases} x, & 0 \le x \le 1 \\ -x, & -1 \le x < 0. \end{cases}$$

Assim $f_p(x) = |x|$. Como $L = 1$, temos

$$a_0 = 2\int_0^1 x dx = x^2\Big|_0^1 = 1.$$

Além disso

$$a_n = 2\int_0^1 x \cdot \cos(\pi n x) dx = \left\{ \begin{array}{c} u' = \cos(\pi n x) \\ v = x \end{array} \right\} = 2\frac{\operatorname{sen}(n\pi x)}{\pi n} \cdot x\Big|_0^1 - \frac{2}{\pi n}\int_0^1 \operatorname{sen}(\pi n x) dx$$

$$= \frac{2}{(\pi n)^2}\cos(\pi n x)\Big|_0^1 = \frac{2}{(\pi n)^2}((-1)^n - 1) = \begin{cases} 0, & n = 2k, \\ -\dfrac{4}{\pi(2k-1)^2}, & n = 2k-1. \end{cases}$$

Portanto

$$S(x) = S[f_p](x) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{-4}{\pi^2(2k-1)^2}\cos((2k-1)\pi x).$$

Agora

$$S(x) = S[f_p](x) = f_p(x) = |x|, \quad x \in (-1,1).$$

Temos

$$S(\pm 1) = S[f_p](\pm 1) = \frac{\lim_{x\to -1+} f_p + \lim_{x\to 1-} f_p}{2} = \frac{1+1}{2} = 1.$$

Então

$$S(x) = |x|, \quad x \in [-1,1].$$

Observe que $S(x)$ e 2-periódica. Logo

$$S(x) = S[f_p](x) = |x - 2n|, \qquad x \in (-1+2n, 1+2n),\ n \in \mathbb{Z}.$$

Temos que

$$S(2012) = S(2014+1) = S(1) = 1.$$

$$S(\frac{-2013}{2}) = S(-1006 - \frac{1}{2}) = S(-\frac{1}{2}) = \frac{1}{2}.$$

;-)

■ **Exemplo 14.3** Usando os resultados do exercício anterior, ache:

$$1)\sum_{k=1}^{\infty} \frac{1}{\pi^2(2k-1)^2}, \qquad 2)\sum_{k=1}^{\infty} \frac{1}{\pi^4(2k+1)^4}.$$

Solução. 1)

$$S(0) = 0 = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{-4}{\pi^2(2k-1)^2},$$

portanto

$$\sum_{k=1}^{\infty} \frac{1}{\pi^2(2k-1)^2} = \frac{1}{8}.$$

2) Usando a Identidade de Parseval, temos

$$\frac{1}{L}\int_{L}^{L} f_p^2 dx = \frac{a_0^2}{2} + \sum_{l=1}^{\infty}(b_n^2 + a_n^2), \qquad f_p^2(x) = x^2.$$

Observe que $L = 1$, $a_0 = 1$, $a_{2k} = 0$, $a_{2k-1} = \dfrac{-4}{\pi^2(2k-1)^2}$, $b_k = 0$. Logo, pela Identidade de Parseval,

$$\int_{-1}^{1} x^2 dx = \frac{a_0^2}{2} + \sum_{k=1}^{\infty} a_{2k-1}^2 = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{16}{\pi^4(2k-1)^4}.$$

Como

$$\int_{-1}^{1} x^2 dx = \frac{x^3}{3}\Big|_{-1}^{1} = \frac{2}{3},$$

obtemos

$$\frac{2}{3} = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{16}{\pi^4(2k-1)^4}.$$

Portanto

$$\sum_{k=1}^{\infty} \frac{1}{\pi^4(2k-1)^4} = \frac{1}{16}\Big(\frac{2}{3} - \frac{1}{2}\Big) = \frac{1}{86}.$$

;-)

■ **Exemplo 14.4** Sabendo que a série de senos de $f(x) = x(\pi - x)$ definida em $[0, \pi]$ é dada por

$$S(x) = S[f_i](x) = \sum_{k=1}^{\infty} \frac{8}{\pi} \frac{1}{(2k-1)^3} \operatorname{sen}((2k-1)x),$$

ache $\sum\limits_{k=1}^{\infty} \dfrac{1}{(2k-1)^6}$.

Solução. Temos $a_0 = a_k = 0$, $b_{2k} = 0$, $b_{2k-1} = \dfrac{8}{\pi(2k-1)^3}$, logo pela Identidade de Parseval,

$$\frac{1}{\pi}\int_{-\pi}^{\pi}(f_i(x))^2dx = \frac{a_0^2}{2} + \sum_{k=1}^{\infty}(a_k^2+b_k^2) = \sum_{k=1}^{\infty}\frac{64}{\pi^2(2k-1)^6}.$$

$$\begin{aligned}\frac{1}{\pi}\int_{-\pi}^{\pi}f_i^2dx &= \frac{2}{\pi}\int_0^{\pi}f^2dx = \frac{2}{\pi}\int_0^{\pi}(x\pi-x^2)^2dx = \frac{2}{\pi}\int_0^{\pi}((x\pi)^2-2\pi x^3+x^4)dx\\ &= \frac{2}{\pi}\Big(\pi^2\frac{x^3}{3}-\frac{2\pi}{4}x^4+\frac{x^5}{5}\Big)\Big|_0^{\pi} = \frac{2}{\pi}\Big(\frac{\pi^3}{3}-\frac{\pi^5}{2}+\frac{\pi^5}{5}\Big) = \frac{2\pi^4}{30}(10-15+6) = \frac{\pi^4}{15}.\end{aligned}$$

Então

$$\frac{\pi^4}{15} = \sum_{k=1}^{\infty}\frac{64}{\pi^2(2k-1)^6}$$

e portanto

$$\sum_{k=1}^{\infty}\frac{1}{(2k-1)^6} = \frac{\pi^4}{15}\cdot\frac{\pi^2}{64} = \frac{\pi^6}{360}.$$

;-)

■ **Exemplo 14.5** 1) Ache $c_1,\dots,c_5$ tais que

$$I = \int_{-\pi}^{\pi}\big[(2x^2-x)-c_1-c_2\cos(x)-c_3\,\mathrm{sen}(x)-c_4\cos(2x)-c_5\,\mathrm{sen}(2x)\big]^2dx$$

tenha o menor valor possível.

2) Ache este valor.

Solução. 1) $c_1,\dots,c_5$ são coeficientes de Fourier da função $f(x) = 2x^2-x$.

$$\begin{aligned}S[f] &= 2S[x^2]-2[x] = 2\cdot\Big(\frac{\pi^2}{3}+\sum_{n=1}^{\infty}\frac{(-1)^n\cdot 4}{n^2}\cos(nx)\Big)-\sum_{n=1}^{\infty}\frac{2\cdot(-1)^{n+1}}{n}\,\mathrm{sen}(nx)\\ &= \frac{2\pi^2}{3}+\sum_{n=1}^{\infty}\Big(\frac{(-1)^n\cdot 8}{n^2}\cos(nx)+\frac{2\cdot(-1)^n}{n}\,\mathrm{sen}(nx)\Big).\end{aligned}$$

Assim

$$\begin{aligned}c_1 &= \frac{2\pi^2}{3} = \frac{a_0}{2},\\ c_2 &= a_1 = -8,\\ c_3 &= b_1 = -2,\\ c_4 &= a_2 = \frac{8}{2^2} = 2,\\ c_5 &= b_2 = \frac{2}{2} = 1.\end{aligned}$$

2)

$$\min_{c_1,\ldots,c_5} I = \int_{-\pi}^{\pi}\Big[(2x^2-x)-\frac{2\pi^2}{3}+8\cos(x)+2\,\mathrm{sen}(x)-2\cos(2x)-\mathrm{sen}(2x)\Big]^2dx$$
$$=\int_{-\pi}^{\pi} f^2dx-2\int_{-\pi}^{\pi} f\cdot S_2[f]dx+\int_{-\pi}^{\pi}\Big(S_2[f]\Big)^2dx.$$

Lembre que

$$\int_{-\pi}^{\pi} f\cdot S_2[f]dx=\int_{-\pi}^{\pi}\Big(S_2[f]\Big)^2dx=\pi\frac{a_0^2}{2}+\pi\sum_1^2(a_k^2+b_k^2).$$

Portanto, temos

$$\min_{c_1,\ldots,c_5} I=\int_{-\pi}^{\pi} f^2dx-\int_{-\pi}^{\pi} f\cdot S_2[f]dx=\int_{-\pi}^{\pi} f^2dx-\pi\Big(\frac{a_0^2}{2}+a_1^2+a_2^2+b_1^2+b_2^2\Big)$$
$$=\int_{-\pi}^{\pi}(2x^2-x)^2dx-\pi\Big(\frac{16\pi^4}{9}\cdot\frac{1}{2}+64+6+6+1\Big)=\Big(\frac{4}{5}x^5+\frac{x^3}{3}\Big)\Big|_{-\pi}^{\pi}-\pi(\frac{8}{9}\pi^4+73)$$
$$=\frac{8}{5}\pi^5+\frac{2}{3}\pi^3-\frac{8}{9}\pi^5-\pi\cdot 73=\frac{72\pi^5+30\pi^3-40\pi^5}{45}-\pi\cdot 73=\frac{\pi^3(32\pi^2+30)}{45}-73\pi.$$

;-)

■ **Exemplo 14.6** Ache a soma de série de Fourier de

$$f(x)=\begin{cases}2, & -2\le x\le -1\\ |x|, & -1<x<1\\ 3, & 1\le x\le 2.\end{cases}$$

Solução. Temos

$$S[f](x)=\begin{cases}2, & -2<x<-1\\ |x|, & -1<x<1\\ 3, & 1<x<2.\end{cases}$$

Agora

$$S[f](-1)=\frac{\lim_{x\to -1+}f(x)+\lim_{x\to -1-}f(x)}{2}=\frac{1+2}{2}=\frac{3}{2},$$
$$S[f](1)=\frac{\lim_{x\to 1+}f(x)+\lim_{x\to 1-}f(x)}{2}=\frac{3+1}{2}=4,$$
$$S[f](\pm 2)=\frac{\lim_{x\to -2+}f(x)+\lim_{x\to 2-}f(x)}{2}=\frac{3+2}{2}=\frac{5}{2}.$$

Como $S[f](x)$ é 4-periódica, obtemos

$$f(x)=\begin{cases}\frac{5}{2}, & x=\pm 2+4n\\ 2, & -2+4n<x<-1+4n,\\ \frac{3}{2}, & x=-1+4n, \qquad n\in\mathbb{Z},\\ |x-4n|, & -1+4n<x<1+4n,\\ 2, & x=1+4n,\\ 3, & 1+4n<x<2+4n.\end{cases}$$